Problem 1.3

Force from a cone

(a) A charge q is located at the tip of a hollow cone (such as an ice cream cone without the ice cream) with surface charge density σ. The slant height of the cone is L, and the half-angle at the vertex is θ. What can you say about the force on the charge q due to the cone?

(b) If the top half of the cone is removed and thrown away (see Fig. 1.31), what is the force on the charge q due to the remaining part of the cone? For what angle θ is this force maximum?

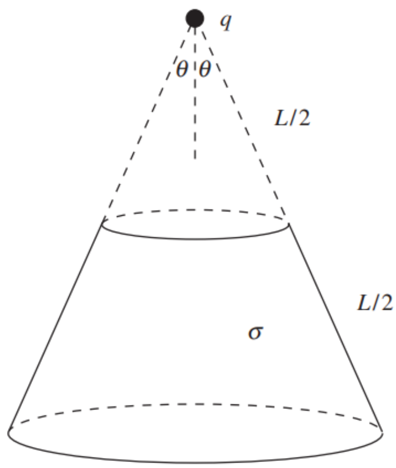
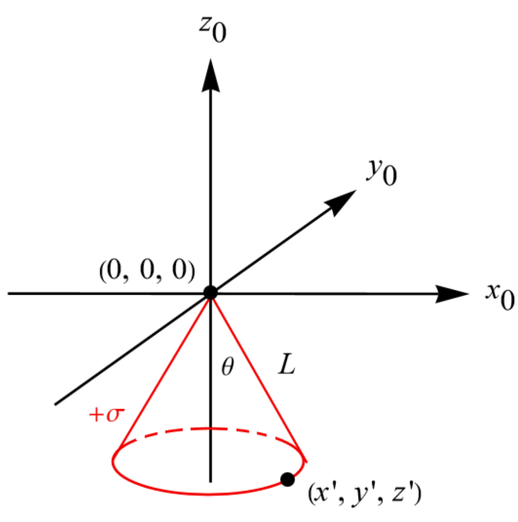


Figure 1.31.

Solution

Using Multivariable Calculus

According to Coulomb's law, the force acting on the charge q from the charged cone is $\mathbf{F} = q\mathbf{E}$, where $\mathbf{E}$ is the electric field of the cone at its vertex. Choose a coordinate system in which the vertex is at the origin, and label some point on the continuous charge distribution as (x', y', z').



The electric field due to a general surface charge distribution is given by

$$\begin{aligned}\mathbf{E}(x,y,z) &= \frac{1}{4\pi\epsilon_0}\iint_{\mathcal{S}} \frac{\sigma(x',y',z')}{\scriptr^2}\hat{\boldsymbol{\scriptr}}\,da' \\ &= \frac{1}{4\pi\epsilon_0}\iint_{\mathcal{S}} \frac{\sigma(x',y',z')}{\scriptr^2}\left(\frac{\boldsymbol{\scriptr}}{\scriptr}\right)da' \\ &= \frac{1}{4\pi\epsilon_0}\iint_{\mathcal{S}} \sigma(x',y',z')\left(\frac{\boldsymbol{\scriptr}}{\scriptr^3}\right)da',\end{aligned}$$

which is essentially the vector sum of electric fields from the individual charges that the surface consists of. $\boldsymbol{\scriptr}$ is the position vector from (x',y',z') to (x,y,z), the point we want to know the electric field at, and $\scriptr$ is its magnitude. For this cone in particular, the charge density σ is a constant and can be pulled in front of the integral.

$$\begin{aligned}\mathbf{E}(x,y,z) &= \frac{\sigma}{4\pi\epsilon_0}\iint_{\mathcal{S}}\left(\frac{\boldsymbol{\scriptr}}{\scriptr^3}\right)da' \\ &= \frac{\sigma}{4\pi\epsilon_0}\iint_{\mathcal{S}} \frac{\langle x-x',y-y',z-z'\rangle}{\left[\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right]^3}\,da' \\ &= \frac{\sigma}{4\pi\epsilon_0}\iint_{\mathcal{S}} \frac{\langle x-x',y-y',z-z'\rangle}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\,da'\end{aligned}$$

Because of the choice to put the cone's vertex at the origin, it's only necessary to evaluate $\mathbf{E}(0,0,0)$. This simplifies the integrand tremendously.

$$\begin{aligned}\mathbf{E}(0,0,0) &= \frac{\sigma}{4\pi\epsilon_0}\iint_{\mathcal{S}} \frac{\langle 0-x',0-y',0-z'\rangle}{[(0-x')^2+(0-y')^2+(0-z')^2]^{3/2}}\,da' \\ &= -\frac{\sigma}{4\pi\epsilon_0}\iint_{\mathcal{S}} \frac{\langle x',y',z'\rangle}{(x'^2+y'^2+z'^2)^{3/2}}\,da' \\ &= -\frac{\sigma}{4\pi\epsilon_0}\left\langle \iint_{\mathcal{S}} \frac{x'}{(x'^2+y'^2+z'^2)^{3/2}}\,da', \iint_{\mathcal{S}} \frac{y'}{(x'^2+y'^2+z'^2)^{3/2}}\,da', \iint_{\mathcal{S}} \frac{z'}{(x'^2+y'^2+z'^2)^{3/2}}\,da' \right\rangle\end{aligned}$$

A cone is most conveniently described using spherical coordinates (r',θ',ϕ'), where r' is the distance from the origin, θ' is the angle from the polar axis, and ϕ' is the azimuthal angle.

$$\begin{cases} x' = r'\sin\theta'\cos\phi' = r'\sin(\pi-\theta)\cos\phi' = r'\sin\theta\cos\phi' \\ y' = r'\sin\theta'\sin\phi' = r'\sin(\pi-\theta)\sin\phi' = r'\sin\theta\sin\phi' \\ z' = r'\cos\theta' = r'\cos(\pi-\theta) = -r'\cos\theta \\ da' = r'\sin(\pi-\theta)\,dr'\,d\phi' = r'\sin\theta\,dr'\,d\phi' \end{cases}$$

As a result,

$$\begin{aligned}
x'^2 + y'^2 + z'^2 &= (r' \sin\theta \cos\phi')^2 + (r' \sin\theta \sin\phi')^2 + (-r' \cos\theta)^2 \\
&= r'^2 \sin^2\theta(\cos^2\phi' + \sin^2\phi') + r'^2 \cos^2\theta \\
&= r'^2 \sin^2\theta + r'^2 \cos^2\theta \\
&= r'^2,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}(0,0,0) &= -\frac{\sigma}{4\pi\epsilon_0} \left\langle \iint\limits_{\mathcal{S}} \frac{x'}{(x'^2 + y'^2 + z'^2)^{3/2}}\, da', \iint\limits_{\mathcal{S}} \frac{y'}{(x'^2 + y'^2 + z'^2)^{3/2}}\, da', \iint\limits_{\mathcal{S}} \frac{z'}{(x'^2 + y'^2 + z'^2)^{3/2}}\, da' \right\rangle \\
&= -\frac{\sigma}{4\pi\epsilon_0} \left\langle \int_0^{2\pi} \int_0^L \frac{r' \sin\theta \cos\phi'}{(r'^2)^{3/2}} (r' \sin\theta\, dr'\, d\phi'), \right. \\
&\qquad \int_0^{2\pi} \int_0^L \frac{r' \sin\theta \sin\phi'}{(r'^2)^{3/2}} (r' \sin\theta\, dr'\, d\phi'), \\
&\qquad \left. \int_0^{2\pi} \int_0^L \frac{-r' \cos\theta}{(r'^2)^{3/2}} (r' \sin\theta\, dr'\, d\phi') \right\rangle \\
&= -\frac{\sigma}{4\pi\epsilon_0} \left\langle \sin^2\theta \int_0^{2\pi} \int_0^L \frac{r'^2 \cos\phi'}{r'^3}\, dr'\, d\phi', \right. \\
&\qquad \sin^2\theta \int_0^{2\pi} \int_0^L \frac{r'^2 \sin\phi'}{r'^3}\, dr'\, d\phi', \\
&\qquad \left. -\cos\theta \sin\theta \int_0^{2\pi} \int_0^L \frac{r'^2}{r'^3}\, dr'\, d\phi \right\rangle \\
&= -\frac{\sigma}{4\pi\epsilon_0} \left\langle \sin^2\theta \left(\int_0^{2\pi} \cos\phi'\, d\phi' \right) \left(\int_0^L \frac{dr'}{r'} \right), \right. \\
&\qquad \sin^2\theta \left(\int_0^{2\pi} \sin\phi'\, d\phi' \right) \left(\int_0^L \frac{dr'}{r'} \right), \\
&\qquad \left. -\frac{1}{2} \sin 2\theta \left(\int_0^{2\pi} d\phi \right) \left(\int_0^L \frac{dr'}{r'}\, dr' \right) \right\rangle \\
&= -\frac{\sigma}{4\pi\epsilon_0} \left\langle (\sin^2\theta)(0) \left(\int_0^L \frac{dr'}{r'} \right), (\sin^2\theta)(0) \left(\int_0^L \frac{dr'}{r'} \right), -\frac{1}{2}(\sin 2\theta)(2\pi) \left(\int_0^L \frac{dr'}{r'}\, dr' \right) \right\rangle.
\end{aligned}$$

The electric field at the cone's vertex is undefined because the integral in dr' is divergent. Therefore, the force on charge q at the tip of the cone is undefined.

If the top half of the cone is removed, then the analysis is the same but the integral in dr' goes from $L/2$ to L instead, resulting in a convergent integral.

$$\begin{aligned}
\mathbf{E}(0,0,0) &= -\frac{\sigma}{4\pi\epsilon_0}\left\langle (\sin^2\theta)(0)\left(\int_{L/2}^{L}\frac{dr'}{r'}\right), (\sin^2\theta)(0)\left(\int_{L/2}^{L}\frac{dr'}{r'}\right), -\frac{1}{2}(\sin 2\theta)(2\pi)\left(\int_{L/2}^{L}\frac{dr'}{r'}\,dr'\right)\right\rangle \\
&= -\frac{\sigma}{4\pi\epsilon_0}\left\langle 0, 0, -\frac{1}{2}(\sin 2\theta)(2\pi)\left(\ln r'\Big|_{L/2}^{L}\right)\right\rangle \\
&= -\frac{\sigma}{4\pi\epsilon_0}\left\langle 0, 0, -\pi\sin 2\theta\left(\ln L - \ln\frac{L}{2}\right)\right\rangle \\
&= -\frac{\sigma}{4\pi\epsilon_0}\left\langle 0, 0, -\pi\sin 2\theta\ln\frac{L}{\frac{L}{2}}\right\rangle \\
&= -\frac{\sigma}{4\pi\epsilon_0}\langle 0, 0, -\pi(\sin 2\theta)(\ln 2)\rangle \\
&= \frac{\sigma\ln 2}{4\epsilon_0}(\sin 2\theta)\langle 0, 0, 1\rangle \\
&= \frac{\sigma\ln 2}{4\epsilon_0}(\sin 2\theta)\hat{\mathbf{z}}_0
\end{aligned}$$

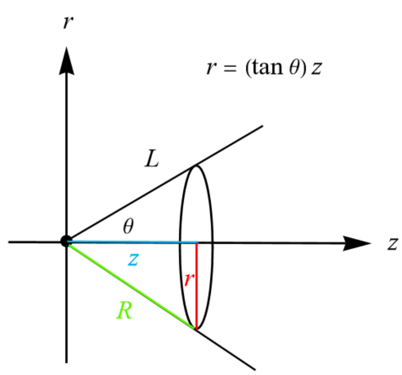
In this case, the force on charge q is

$$\mathbf{F} = \frac{q\sigma\ln 2}{4\epsilon_0}(\sin 2\theta)\hat{\mathbf{z}}_0,$$

which is maximum when $\sin 2\theta = 1$, that is, when $\theta = \pi/4$.

Using Single-Variable Calculus

Because the cone and its charge distribution are symmetric about an axis, the electric field may be determined with single-variable calculus. This is done by recognizing that the electric field of the cone is due to the vector sum of electric fields from the many rings the cone consists of.



$$\mathbf{E}_{\text{cone}} = \int d\mathbf{E}_{\text{ring}} = \frac{-\hat{\mathbf{z}}\cos\theta}{4\pi\epsilon_0}\int\frac{dQ_{\text{ring}}}{R^2} = \frac{-\hat{\mathbf{z}}\cos\theta}{4\pi\epsilon_0}\int\frac{\sigma(z)\,dS}{r^2+z^2} = \frac{-\hat{\mathbf{z}}\cos\theta}{4\pi\epsilon_0}\int\frac{\sigma\,[2\pi r(z)\,ds]}{[r(z)]^2+z^2}$$

Write the integral in dz by using the formula for arc length.

$$\begin{aligned}
\mathbf{E}_{\text{cone}} &= \frac{-\sigma\hat{\mathbf{z}}\cos\theta}{2\epsilon_0}\int \frac{r(z)}{[r(z)]^2+z^2}\,ds \\
&= \frac{-\sigma\hat{\mathbf{z}}\cos\theta}{2\epsilon_0}\int_0^{L\cos\theta} \frac{r(z)}{[r(z)]^2+z^2}\sqrt{1+\left(\frac{dr}{dz}\right)^2}\,dz \\
&= \frac{-\sigma\hat{\mathbf{z}}\cos\theta}{2\epsilon_0}\int_0^{L\cos\theta} \frac{z\tan\theta}{z^2\tan^2\theta+z^2}\sqrt{1+\tan^2\theta}\,dz \\
&= \frac{-\sigma\hat{\mathbf{z}}\cos\theta}{2\epsilon_0}\int_0^{L\cos\theta} \frac{\tan\theta}{z(\tan^2\theta+1)}\sec\theta\,dz \\
&= \frac{-\sigma\hat{\mathbf{z}}}{2\epsilon_0}\int_0^{L\cos\theta} \frac{\tan\theta}{z(\sec^2\theta)}\,dz \\
&= \frac{-\sigma\hat{\mathbf{z}}}{2\epsilon_0}\int_0^{L\cos\theta} \frac{\sin\theta\cos\theta}{z}\,dz \\
&= \frac{-\sigma\hat{\mathbf{z}}}{2\epsilon_0}\int_0^{L\cos\theta} \frac{\frac{1}{2}\sin 2\theta}{z}\,dz \\
&= \frac{-\sigma\hat{\mathbf{z}}}{4\epsilon_0}(\sin 2\theta)\int_0^{L\cos\theta} \frac{dz}{z}
\end{aligned}$$

This integral diverges, so the electric field is undefined at the origin. If the left half of the cone is removed, the analysis is the same except for the limits of integration.

$$\begin{aligned}
\mathbf{E}_{\text{cone}} &= \frac{-\sigma\hat{\mathbf{z}}}{4\epsilon_0}(\sin 2\theta)\int_{L\cos\theta/2}^{L\cos\theta} \frac{dz}{z} \\
&= \frac{-\sigma\hat{\mathbf{z}}}{4\epsilon_0}(\sin 2\theta)\ln\frac{L\cos\theta}{\frac{L\cos\theta}{2}} \\
&= \frac{-\sigma\hat{\mathbf{z}}}{4\epsilon_0}(\sin 2\theta)(\ln 2)
\end{aligned}$$

The force on q is then

$$\mathbf{F} = \frac{-q\sigma\hat{\mathbf{z}}}{4\epsilon_0}(\sin 2\theta)(\ln 2).$$

As before, the force is largest when $\sin 2\theta = 1$, that is, when $\theta = \pi/4$.